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Approximations to flow in slender tubes

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Summary

A tube of circular cross section whose radius is a function of a slow variable $Z=(1/R)z$, where z is the co-ordinate in the axial direction and R is a large streamwise Reynolds number, may be designated a slender tube. An elementary approximation to the flow in such tubes is obtained and results compared with an approximation based on the profiles obtained by Daniels and Eagles [7] for exponential slender tubes.

1. Introduction

The steady flow in "slender tubes" will be studied in this paper. We take dimensionless cylindrical polar co-ordinates (r, θ, z) and suppose the rigid wall of the tube has the equation

$$r=H(Z), \quad \text{with } Z=\epsilon z,$$

where ϵ is a small parameter, and we assume that the streamwise Reynolds number R, defined in (2.1), is related to ϵ by

$$\epsilon R=\lambda=O(1), \quad \text{as } \epsilon\to 0.$$

Thus the scale of variation in the axial direction is proportional to R. It can then be shown that if the Stokes stream function ψ is expanded in the form $\psi=\psi_0(r, Z)+\epsilon^2\psi_1(r, Z)+\ldots$, the equation for ψ_0 is essentially the classical boundary-layer equation. The problem for ψ_0 is called the slender-tube problem.

There has been much recent work on the corresponding slender channels, see for example Blottner [1], Eagles and Smith [2], Plotkin [3] and Allmen [4]. Some interesting flows can be calculated, including some with separation and re-attachment of the main stream. We mention also the work of Fraenkel [5,6] in which he shows that for a particular class of slender channels with very small curvature of the walls the solution may be expanded systematically in such a way that the appropriate local Jeffery-Hamel profile is the first approximation at each station.

The case of slender tubes is slightly more difficult because of the extra complexity of the equations of motion and the singularity in these equations at $r=0$. It is, of course, true that the general slender-tube problem may be tackled by finite-difference methods of integrating the boundary-layer equations. However, the computing work is not easy, involving the repeated solution of non-linear systems of equations, so in this paper we develop an elementary method of approximation. The solution is expanded in powers of λ multiplied by functions of Z and η, for which explicit forms are obtained up to the term in λ^3. At first sight it might be thought that such a series would be of very limited value, being useful only for small values of λ. However, the nature of the series indicates it is probably *convergent* for a range of λ, rather than merely asymptotic (for $\lambda \to 0$), and numerical calculations indicate that in many cases it is highly plausible that the series is useful for surprisingly large values of λ.

Daniels and Eagles [7] considered the special case in which $H(Z) = \exp(aZ)$ and showed that the flow was governed by an ordinary differential equation with independent variable $\eta = r/H(Z)$, which allows interesting solutions. The branch 1 family of such streamwise velocity profiles which are most likely to be physically relevant include profiles with points of inflexion, drastically different from Poiseuille flow. It is a one-parameter family, the parameter being $\gamma = \lambda H'(Z)/H(Z) = \lambda a$. We shall hereinafter refer to these profiles as the DE profiles.

Eagles [8] generalized this work to tubes which are 'locally exponential' in the sense that $(\mathrm{d}H/\mathrm{d}Z)/H = f(\epsilon Z)$, and showed that for many such tubes the DE profiles provide an excellent first approximation, the appropriate DE profile with $\gamma = \lambda H'(Z)/H(Z)$ appearing as the first term in a systematic expansion for small ϵ, and higher-order terms being shown to be numerically very small. This work is similar to that of Fraenkel [5,6] for channels. Eagles conjectured that the DE profiles were likely to be a good approximation in more general tubes.

Using our series in powers of λ, we are able to expand the difference between an exact solution and the local profile in powers of λ. This series starts with λ^2, and numerical results are presented which make it plausible that if certain parameters are of limited size the local DE profiles, with $\gamma = \lambda H'(Z)/H(Z)$ at each station, are indeed a good approximation to the flow, containing all the essential qualitative features, and being numerically close to the exact solution in many cases.

2. Expansion method for flow in slender tubes

We take (r, θ, z) as cylindrical polar co-ordinates, r and z being made dimensionless by the radius L of the tube at $z = 0$. Then we define the Reynolds number

$$R = \frac{M}{\nu L} \tag{2.1}$$

where M is the volumetric flow rate and ν is the kinematic viscosity.

We suppose flow is taking place in a tube whose rigid boundary is given by

$$r = H(Z) \tag{2.2}$$

where

$$Z = \epsilon z. \tag{2.3}$$

We also assume that

$$R\epsilon = \lambda = \mathrm{O}(1) \tag{2.4}$$

as $\epsilon \to 0$. The Stokes stream function, made dimensionless by M, satisfies the equation

$$\frac{1}{r^2}\left(\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}-\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\right)D^2\psi+\frac{1}{r^3}\left(2\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial z^2}+3\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial r^2}-\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial r\,\partial z}\right)-\frac{3}{r^4}\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial r}$$

$$=\frac{1}{R}\left\{\frac{1}{r}D^4\psi-\frac{2}{r^2}\left(\frac{\partial^3\psi}{\partial r\,\partial z^2}+\frac{\partial^3\psi}{\partial r^3}\right)+\frac{3}{r^3}\frac{\partial^2\psi}{\partial r^2}-\frac{3}{r^4}\frac{\partial\psi}{\partial r}\right\} \tag{2.5}$$

where

$$D\equiv\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}. \tag{2.6}$$

Now if $\partial\psi/\partial z=O(\epsilon)$, $\partial^2\psi/\partial z^2=O(\epsilon^2)$ etc., and $R=O(1/\epsilon)$ the dominant terms are of order ϵ. It is appropriate to expand the stream function in the form

$$\psi=F(\eta,\,Z)+\epsilon^2Q(n,\,Z)+\ldots \tag{2.7}$$

where

$$\eta=r/H(Z) \tag{2.8}$$

is the local cross-stream variable. Then we find the equation for F is

$$L(F)=\lambda\left[4\frac{H'(Z)}{H(Z)}\left(\frac{1}{\eta^2}\left\{\frac{\partial F}{\partial\eta}\right\}^2-\frac{1}{\eta}\frac{\partial F}{\partial\eta}\frac{\partial^2F}{\partial\eta^2}\right)+\frac{\partial F}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial^3F}{\partial\eta^2\,\partial Z}-\frac{1}{\eta^2}\frac{\partial^2F}{\partial\eta\,\partial Z}\right)\right.$$

$$\left.+\frac{\partial F}{\partial Z}\left(-\frac{1}{\eta}\frac{\partial^3F}{\partial\eta^3}+\frac{3}{\eta^2}\frac{\partial^2F}{\partial\eta^2}-\frac{3}{\eta^3}\frac{\partial F}{\partial\eta}\right)\right] \tag{2.9}$$

where

$$L\equiv\frac{\partial^4}{\partial\eta^4}-\frac{2}{\eta}\frac{\partial^3}{\partial\eta^3}+\frac{3}{\eta^2}\frac{\partial^2}{\partial\eta^2}-\frac{3}{\eta^3}\frac{\partial}{\partial\eta}. \tag{2.10}$$

This is a form of the boundary-layer equation. (If the lengths were non-dimensionalized by a length LR instead of L the scalings would appear in more familiar form.)

It should be noted that the dimensionless fluid velocities in the r and z directions respectively are, within this approximation,

$$u=-r^{-1}\frac{\partial\psi_0}{\partial z}=\epsilon\left(\frac{-1}{\eta H}\frac{\partial F}{\partial Z}+\frac{H'}{H^2}\frac{\partial F}{\partial\eta}\right), \tag{2.11}$$

$$v=r^{-1}\frac{\partial\psi_0}{\partial r}=\frac{1}{\eta H^2}\frac{\partial F}{\partial\eta} \tag{2.12}$$

and the dimensional fluid velocities are

$$u^* = \frac{M}{L^2}u, \quad v^* = \frac{M}{L^2}v. \tag{2.13}$$

The boundary conditions are that

$$F = \mathrm{O}(\eta^2) \quad \text{as } \eta \to 0, \tag{2.14a}$$

$$\frac{\partial F}{\partial \eta} = 0 \quad \text{when } \eta = 1, \tag{2.14b}$$

$$F = \frac{1}{2\pi} \quad \text{when } \eta = 1. \tag{2.14c}$$

See Daniels and Eagles [7] for further details.

In the cases considered here we shall take $H'(Z) \to 0$ as $Z \to \pm\infty$ and the appropriate flow as $Z \to \pm\infty$ is Poiseuille flow: $F \to (1/\pi)(\eta^2 - \frac{1}{2}\eta^4)$. If equation (2.9) was integrated forward from $Z = -\infty$ starting with Poiseuille flow the solution would ultimately settle down again to Poiseuille flow as $Z \to +\infty$, as found in the analogous channel-flow case by Eagles and Smith [2]. The method to be adopted here will automatically ensure Poiseuille flow at $Z = \pm\infty$ and so would not be suitable if some other profile were imposed at $Z = -\infty$, say.

We expand

$$F(\eta, Z; \lambda) = F^{(0)}(\eta, Z) + \lambda F^{(1)}(\eta, Z) + \lambda^2 F^{(2)}(\eta, Z) + \dots, \tag{2.15}$$

and substituting this into (2.9) yields equations, from coefficients of powers of λ, as follows:

$$L\{F^{(0)}\} = 0, \tag{2.16}$$

$$L\{F^{(1)}\} = 4\frac{H'(Z)}{H(Z)}\left(\frac{1}{\eta^2}\{F_\eta^{(0)}\}^2 - \frac{1}{\eta}F_\eta^{(0)}F_{\eta\eta}^{(0)}\right) + F_\eta^{(0)}\left(\frac{1}{\eta}F_{\eta\eta Z}^{(0)} - \frac{1}{\eta^2}F_{\eta Z}^{(0)}\right)$$
$$+ F_Z^{(0)}\left(-\frac{1}{\eta}F_{\eta\eta\eta}^{(0)} + \frac{3}{\eta^2}F_{\eta\eta}^{(0)} - \frac{3}{\eta^3}F_\eta^{(0)}\right), \tag{2.17}$$

$$L\{F^{(2)}\} = 4\frac{H'(Z)}{H(Z)}\left[F_\eta^{(0)}\left(\frac{2}{\eta^2}F_\eta^{(1)} - \frac{1}{\eta}F_{\eta\eta}^{(1)}\right) - \frac{1}{\eta}F_\eta^{(1)}F_{\eta\eta}^{(0)}\right]$$
$$+ \sum_{j=0}^{1} F_\eta^{(1-j)}\left(\frac{1}{\eta}F_{\eta\eta Z}^{(j)} - \frac{1}{\eta^2}F_{\eta Z}^{(j)}\right)$$
$$+ \sum_{j=0}^{1} F_Z^{(1-j)}\left(-\frac{1}{\eta}F_{\eta\eta\eta}^{(j)} + \frac{3}{\eta^2}F_{\eta\eta}^{(j)} - \frac{3}{\eta^3}F_\eta^{(j)}\right), \tag{2.18}$$

$$L\{F^{(3)}\}=4\frac{H'(Z)}{H(Z)}\left[F_\eta^{(0)}\left(\frac{2}{\eta^2}F_\eta^{(2)}-\frac{1}{\eta}F_{\eta\eta}^{(2)}\right)+F_\eta^{(1)}\left(\frac{1}{\eta^2}F_\eta^{(1)}-\frac{1}{\eta}F_{\eta\eta}^{(1)}\right)-\frac{1}{\eta}F_\eta^{(2)}F_{\eta\eta}^{(0)}\right]$$

$$+\sum_{j=0}^{2}F_\eta^{(2-j)}\left(\frac{1}{\eta}F_{\eta\eta Z}^{(j)}-\frac{1}{\eta^2}F_{\eta Z}^{(j)}\right)$$

$$+\sum_{j=0}^{2}F_Z^{(2-j)}\left(-\frac{1}{\eta}F_{\eta\eta\eta}^{(j)}+\frac{3}{\eta^2}F_{\eta\eta}^{(j)}-\frac{3}{\eta^3}F_\eta^{(j)}\right). \tag{2.19}$$

The solution for $F^{(0)}$ with boundary conditions (2.14) is easily found to be

$$F^{(0)}=\frac{1}{\pi}(\eta^2-\tfrac{1}{2}\eta^4), \tag{2.20}$$

the local Poiseuille-flow solution.

Now the boundary conditions on the functions $F^{(n)}$ for $n>0$ are

$$F^{(n)}=O(\eta^2) \quad \text{as } \eta\to 0, \tag{2.21a}$$

$$F_\eta^{(n)}=0 \quad \text{when } \eta=1, \tag{2.21b}$$

$$F^{(n)}=0 \quad \text{when } \eta=1, \tag{2.21c}$$

and equation (2.17) has a solution of the form

$$F^{(1)}=A(Z)+B(Z)\eta^2+C(Z)\eta^4+E(Z)\eta^2\log\eta+P(\eta,Z)$$

where $P(\eta, Z)$ is a particular integral. The boundary condition (2.21a) eliminates terms independent of η and in $\eta^2\log\eta$. The solution is

$$F^{(1)}(\eta,Z)=\frac{1}{9\pi^2}\frac{H'(Z)}{H(Z)}F_1^{(1)}(\eta) \tag{2.22}$$

where

$$F_1^{(1)}(\eta)=\eta^2-\tfrac{9}{4}\eta^4+\tfrac{3}{2}\eta^6-\tfrac{1}{4}\eta^8. \tag{2.23}$$

We proceed in a similar way to find

$$F^{(2)}(\eta,Z)=\frac{331}{10\,800\pi^3}\left\{\frac{H'(Z)}{H(Z)}\right\}^2F_1^{(2)}(\eta)-\frac{13}{1800\pi^3}\frac{\mathrm{d}}{\mathrm{d}Z}\left\{\frac{H'(Z)}{H(Z)}\right\}F_2^{(2)}(\eta), \tag{2.24}$$

where

$$F_1^{(2)}(\eta)=\eta^2-\tfrac{980}{331}\eta^4+\tfrac{1100}{331}\eta^6-\tfrac{600}{331}\eta^8+\tfrac{165}{331}\eta^{10}-\tfrac{16}{331}\eta^{12}, \tag{2.25}$$

$$F_2^{(2)}(\eta)=\eta^2-\tfrac{145}{52}\eta^4+\tfrac{75}{26}\eta^6-\tfrac{75}{52}\eta^8+\tfrac{15}{39}\eta^{10}-\tfrac{1}{26}\eta^{12}. \tag{2.26}$$

Considering the equation (2.19) for $F^{(3)}$ we see that the products of terms on the right-hand side lead to functions of Z occurring in the forms S^3, SS' and S'' where $S = H'/H$. But since $S'' = 2SS' - (\mathrm{d}/\mathrm{d}Z)(H''/H)$ it is convenient to write the result in the form

$$F^{(3)}(\eta, Z) = \frac{2759}{297\,675\pi^4}\{S(Z)\}^3 F_1^{(3)}(\eta) + \frac{281\,833}{38\,102\,400\pi^4} S(Z)S'(Z)F_2^{(3)}(\eta)$$

$$+ \frac{1459}{3175\,200\pi^4} T(Z)F_3^{(3)}(\eta), \tag{2.27}$$

where

$$S(Z) = H'(Z)/H(Z), \quad T(Z) = (\mathrm{d}/\mathrm{d}Z)\{H''(Z)/H(Z)\} \tag{2.28}$$

and

$$F_1^{(3)}(\eta) = \eta^2 - 3.36616\eta^4 + 4.81353\eta^6 - 4.00156\eta^8 + 2.13121\eta^{10}$$
$$-0.68931\eta^{12} + 0.12059\eta^{14} - 0.00829\eta^{16},$$

$$F_2^{(3)}(\eta) = \eta^2 - 3.07959\eta^4 + 3.88164\eta^6 - 2.80526\eta^8 + 1.34308\eta^{10}$$
$$-0.40371\eta^{12} + 0.06855\eta^{14} - 0.00472\eta^{16},$$

$$F_3^{(3)}(\eta) = \eta^2 - 3.01182\eta^4 + 3.65233\eta^6 - 2.49786\eta^8 + 1.13348\eta^{10}$$
$$-0.32745\eta^{12} + 0.05518\eta^{14} - 0.00386\eta^{16}.$$

It was found to be impracticable to keep rational numbers in the coefficients here.

It should be noted that the way in which we have arranged our results means that in the special case of exponential tubes where $H'(Z)/H(Z) = \text{const.}$ the coefficients of $F_2^{(2)}(\eta)$, $F_2^{(3)}(\eta)$ and $F_3^{(3)}(\eta)$ are zero. The remaining terms constitute an expansion in powers of $\lambda H'(Z)/H(Z)$ of the branch 1 solutions of the DE equation (4.1) with $\gamma = \lambda H'/H$.

Extensive checks were made on the accuracy of the analysis. One method of checking was to fix η, at say 0.5, and calculate numerically both sides of equations (2.19) from (2.20), (2.22), and (2.24).

3. Some numerical results and discussion

An inspection of the terms in the series shows that the coefficients of λ^n decrease rather rapidly with increasing n, and are numerically small. Thus the series might have a large range of usefulness. Since the highest derivative lies with the unknown function at each stage, the series has the general nature of a convergent series rather than merely asymptotic. Thus, for a particular value of λ, if the moduli of the terms are decreasing steadily with n and if the last term is small, we may expect to have a good approximation.

To illustrate we consider two tubes whose radii are

$$H_1(Z) = 1 + \tfrac{1}{2}\tanh Z \tag{3.1}$$

and

$$H_2(Z) = 1 + \tfrac{1}{2}\tanh^2 Z \tag{3.2}$$

which we shall call, respectively, tube 1 and tube 2. The first tube is divergent and the second convergent for $Z < 0$ and divergent for $Z > 0$.

We consider the fluid velocity in the axial direction, in the form

$$G(\eta, Z) = \frac{1}{\eta}\frac{\partial F}{\partial \eta}$$

$$= G^{(0)}(\eta) + \lambda G^{(1)}(\eta, Z) + \lambda^2 G^{(2)}(\eta, Z) + \lambda^3 G^{(3)}(\eta, Z) + \dots \tag{3.3}$$

In Table 1, for tube 1 with $\lambda = 5$, we show the separate contributions of the different powers of λ at selected values of η and Z. We note that λ^3 contribution is less than about $\frac{2}{3}$% of the leading term, and generally is much less than this. Also the ratio of the λ^3 term to the λ^2 term is in general less than $\frac{1}{3}$. Overall, the results give the strong impression of convergence, and the expectation that higher-order contributions would be very small. As $Z \to \pm\infty$, the flow approaches Poiseuille flow.

We might expect useful results for even higher values of λ. The profiles here differ considerably from Poiseuille flow. For example, with $\lambda = 5$, at $Z = 0$ the maximum velocity is about 10% higher than Poiseuille flow, giving a 'sharper' profile with varying curvature. The second derivative of the velocity with respect to η at the centre of the tube is about -1.89 compared with -1.27 for Poiseuille flow, at $Z = 0$.

Table 1. The contributions of separate powers of λ to the axial velocity profile for the tube with $H = 1 + (1/2)\tanh Z$ when $\lambda = 5$

Z	$\lambda H'/H$	η	$G^{(0)}$	$\lambda G^{(1)}$	$\lambda^2 G^{(2)}$	$\lambda^3 G^{(3)}$
−0.8	2.092	0	0.637	0.0471	0.0042	−0.0012
		0.3	0.579	0.0297	0.0022	−0.0007
		0.7	0.325	0.0114	−0.0009	−0.0003
−0.4	2.640	0	0.637	0.0595	0.0124	0.0013
		0.3	0.579	0.0375	0.0067	0.0006
		0.7	0.325	0.0140	−0.0028	−0.0002
0.0	2.500	0	0.637	0.0563	0.0153	0.0042
		0.3	0.579	0.0355	0.0083	0.0021
		0.7	0.325	−0.0136	−0.0035	−0.0009
0.4	1.798	0	0.637	0.0405	0.0111	0.0035
		0.3	0.579	0.0255	0.0061	0.0018
		0.7	0.325	−0.0098	−0.0025	−0.0007
0.8	1.049	0.0	0.637	0.0236	0.0059	0.0017
		0.3	0.579	0.0149	0.0033	0.0009
		0.7	0.325	−0.0057	−0.0014	−0.0004

Table 2. The contributions of separate powers of λ to the axial velocity profile for the tube with $H = 1 + (1/2)\tanh^2 Z$ when $\lambda = 5$

Z	$\lambda H'/H$	η	$G^{(0)}$	$\lambda G^{(1)}$	$\lambda^2 G^{(2)}$	$\lambda^3 G^{(3)}$
−0.6	−1.670	0.0	0.637	−0.0376	0.0058	−0.0004
		0.3	0.579	−0.0237	0.0032	−0.0002
		0.7	0.325	+0.0091	−0.0013	+0.0001
−0.2	−0.930	0.0	0.637	−0.0209	−0.0076	0.0036
		0.3	0.579	−0.0132	−0.0043	0.0019
		0.7	0.325	+0.0051	0.0017	−0.0008
+0.2	0.930	0.0	0.637	0.0209	−0.0076	−0.0036
		0.3	0.579	0.0132	−0.0043	−0.0019
		0.7	0.325	−0.0051	+0.0017	0.0008
0.6	1.670	0.0	0.637	0.0376	0.0058	0.0004
		0.3	0.579	0.0237	0.0032	0.0002
		0.7	0.325	−0.0091	−0.0013	−0.0001

In Table 2, for tube 2 with $\lambda = 5$ and selected values of Z and η, we show the separate contributions of the powers of λ. Again the impression is of rapid convergence and we would expect higher-order terms to be negligible. This is true overall for this case. Here the values of $\lambda H'(Z)/H(Z)$ are smaller than in tube 1, so the flow is closer to Poiseuille flow. Nevertheless it is significantly different. There is a noticeable 'flattening' of the profiles in the convergent part of the tube and a 'sharpening' in the divergent part. Again, we could still expect reasonably accurate results for higher values of λ.

In Figure 1 we show examples of the velocity profiles calculated using the series for tube 2 with $\lambda = 10$. The profiles represent the velocity with respect to η at various values

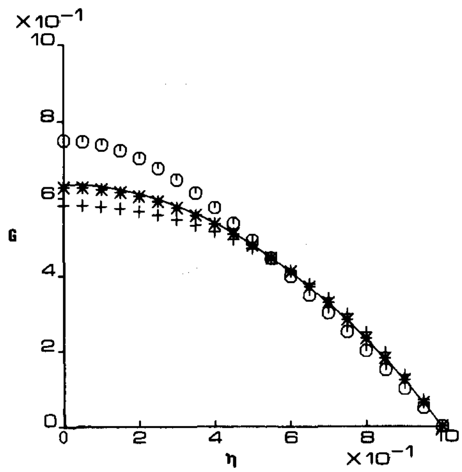

Figure 1. Velocity profile for tube at various values of Z when $\lambda = 10$. – Poiseuille flow; * $Z = -2$; + $Z = -0.6$; ○ $Z = 0.8$.

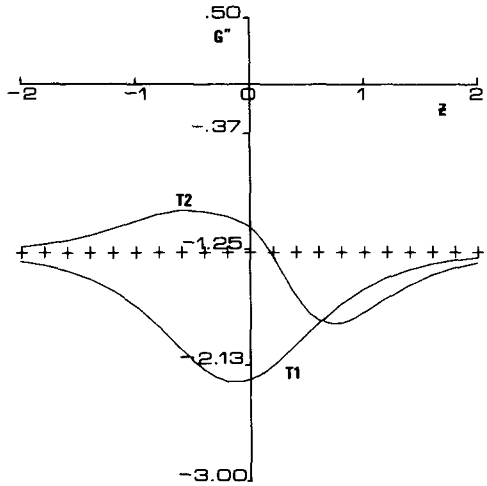

Figure 2. Second derivative of velocity profile as a function of Z at $\eta = 0$ when $\lambda = 6$. T1: Tube 1. T2: Tube 2. + Poiseuille Flow.

of Z in the range $Z = -2$ to $Z = +2$. Figure 2 shows the second derivative of the axial velocity profile as a function of Z at $\eta = 0$ when $\lambda = 6$.

4. Comparison with the Daniels and Eagles profiles

The DE profiles are exact solutions for the slender-tube equations when $\lambda H'/H = \gamma =$ const. The velocity function $g(\eta)$ satisfies the ordinary differential equation

$$g''' - \eta^{-1}g'' - \eta^{-2}g' + 4\gamma g g' = 0 \tag{4.1}$$

with the boundary conditions

$$g(\eta) \text{ is regular at } \eta = 0, \tag{4.2}$$

$$\int_0^1 \eta g(\eta)\, \mathrm{d}\eta = \frac{1}{2\pi}, \quad g(1) = 0. \tag{4.3}$$

In more general tubes, with $(\mathrm{d}H/\mathrm{d}Z)/H = f(\epsilon Z)$ they have been shown to be the first term in an asymptotic series in powers of ϵ, the value of γ in (4.1) being taken as $H'(Z)/H(Z)$ at each value of Z. Higher-order terms were shown to be numerically small, leading Eagles [8] to conjecture that the DE profiles are a good approximation in more general slender tubes.

From our expansion in powers of λ it can be seen that the difference between an exact solution and the DE profile of each value of Z is

$$E(\lambda, \eta, Z) = \lambda^2 \frac{13}{1800\pi^3} \frac{\mathrm{d}S}{\mathrm{d}Z} G_2^{(2)}(\eta) + \lambda^3 \frac{281\,833}{38\,102\,400\pi^4} S \frac{\mathrm{d}S}{\mathrm{d}Z} G_2^{(3)}(\eta)$$

$$+ \lambda^3 \frac{1459}{3175\,200\pi^4} \frac{\mathrm{d}}{\mathrm{d}Z}\left(\frac{H''}{H}\right) G_3^{(3)}(\eta) + \ldots, \tag{4.3}$$

Table 3. Differences between the DE approximation and exact solution. Contributions of separate terms $\lambda^2 E^{(2)}$ and $\lambda^3 E^{(3)}$ for tube with $H = 1 + (1/2)\tanh Z$ when $\lambda = 3$

Z	$\lambda H'/H$	η	DE	$\lambda^2 E^{(2)}$	$\lambda^3 E^{(3)}$
−1.0	1.017	0.0	0.66185	−0.00168	−0.00047
		0.8	0.22184	0.00040	0.00010
−0.4	1.584	0.0	0.67817	−0.00051	−0.00047
		0.8	0.21726	0.00012	0.00010
0.4	1.078	0.0	0.66348	0.00169	0.00051
		0.8	0.22137	−0.00040	−0.00011
1.0	0.456	0.0	0.64732	0.00107	0.00023
		0.8	0.22602	−0.00026	−0.00005

where

$$G_m^{(\eta)}(\eta) = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(F_m^{(\eta)}(\eta)\right) \quad \text{and } S = \frac{H'}{H}. \tag{4.4}$$

We have calculated the separate terms in this series numerically for tube 1 and show some of the results in Table 3 for the case when $\lambda = 3$ for illustration. We denote the series for the error by

$$\lambda^2 E^{(2)}(\eta, Z) + \lambda^3 E^{(3)}(\eta, Z) + \dots.$$

It is apparent that for these tubes with $\lambda = 3$ the DE profiles are an extremely good approximation. Even with $\lambda = 5$ the maximum difference between the velocity (at $\eta = 0$) of the DE profile and the exact solution can be estimated to be less than about 1.5%, and generally much less. The slightly anomalous case in Table 4 at $Z = -0.4$ where the λ^3 contribution is as high as the λ^2 contribution arises from the fact that $E^{(2)}(\eta, Z)$ contains dS/dZ as a factor and this happens to be numerically small for negative Z of moderate size. It should not be taken as evidence of the non-convergence of the series. The case of $Z = 0.4$ should be taken as a better indication of the convergent nature of the series.

Table 4. Values of the velocity functions $G_1^{(1)}$, $G_1^{(2)}$, $G_2^{(2)}$

η	$G_1^{(1)}$	$G_1^{(2)}$	$G_2^{(2)}$
0.0	2.000	2.000	2.000
0.1	1.910	1.883	1.890
0.2	1.655	1.557	1.581
0.3	1.261	1.085	1.129
0.4	0.782	0.559	0.614
0.5	0.281	0.078	0.128
0.6	−0.167	−0.276	−0.249
0.7	−0.484	−0.451	−0.459
0.8	−0.598	−0.440	−0.478
0.9	−0.448	−0.273	−0.316
1.0	0.000	0.000	0.000

5. Further discussion

The dominant contribution to the change from Poiseuille flow is given by $\lambda(H'/H)G_1^{(1)}(\eta)$. In Table 4 we show values of $G_1^{(1)}(\eta)$, $G_1^{(2)}(\eta)$ and $G_2^{(2)}(\eta)$. It can be seen clearly how the function $\lambda(H'/H)G_1^{(1)}(\eta)$ contributes to the flattening of the profiles when $H'/H<0$ and to the sharpening when $H'/H>0$. In fact for the cases considered earlier just the approximation $G^{(0)}+\lambda G^{(1)}$ seems to give a remarkably good approximation for $\lambda<3$.

We have proposed two possible approximations. The first one is the use of the series $G^{(0)}+\lambda G^{(1)}+\lambda^2G^{(2)}+\lambda^3G^{(3)}$ for the axial velocity. By using the expressions calculated earlier we can show that

$$G^{(3)}(0, Z) \doteq 0.943\times10^{-5}T(Z)-0.1519\times10^{-3}S(Z)S'(Z)$$
$$+0.1903\times10^{-3}\{S(Z)\}^3 \qquad (5.1)$$

where $T(Z)=(\mathrm{d}/\mathrm{d}Z)(H''/H)$ and $S(Z)=H'/H$. Ignoring the first term we propose that a practical rule for the application of this approximation is that

$$\lambda^3\,|0.029S^3-0.024SS'|<1.$$

The λ^3 term is then less than about 1% of the leading term at $\eta=0$. As an example, for tube 1 at $Z=0$ with $S(0)=\frac{1}{2}$ and $S'(0)=-\frac{1}{4}$ this gives $\lambda<5.3$.

The second proposed approximation is to use the local DE profile at each stage. The leading term in the correction to this approximation is calculated from (4.3) to be $-0.0004658\lambda^2S'$ at $\eta=0$. We suggest that a practical rule for the usefulness of the DE approximation is that the modulus of this term should be less than 0.005. This leads to

$$\lambda^2\,|S'(Z)|<10,$$

approximately. As an example for tube 1 at $Z=0$ this requires $\lambda<6.3$.

We conclude by mentioning the question of the stability of these flows. It is known that Poiseuille flow is stable to small disturbances for all Reynolds numbers. We believe that many slender-channel flows are unstable for sufficiently high values of the parameter $\lambda H'(Z)/H(Z)$. The present approximation makes a study of their stability more easily possible. This study is under way and results will be presented at a later date.

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